

# Plancherel Formulae associated to Filtrations of Nilpotent Lie Groups

Joseph A. Wolf

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## Abstract

We study the conditions for a nilpotent Lie group to be foliated into subgroups that have square integrable (relative discrete series) unitary representations, that fit together to form a filtration by normal subgroups. Then we use that filtration to construct a class of “stepwise square integrable” representations on which Plancherel measure is concentrated. Further, we work out the character formulae for those stepwise square integrable representations, and we give an explicit Plancherel formula. Next, we use some structure theory to check that all these constructions and results apply to nilradicals of minimal parabolic subgroups of real reductive Lie groups. Finally, we develop multiplicity formulae for compact quotients  $N/\Gamma$  where  $\Gamma$  respects the filtration.

## 1 Introduction

There is a well developed theory of square integrable representations of nilpotent Lie groups [8]. It is, of course, based on the general representation theory [6] for nilpotent Lie groups. A connected simply connected Lie group  $N$  with center  $Z$  is called *square integrable* if it has unitary representations  $\pi$  whose coefficients  $f_{u,v}(x) = \langle u, \pi(x)v \rangle$  satisfy  $|f_{u,v}| \in L^2(N/Z)$ . If  $N$  has one such square integrable representation then there is a certain polynomial function  $\text{Pf}(\lambda)$  on the linear dual space  $\mathfrak{z}^*$  of the Lie algebra of  $Z$  that is key to harmonic analysis on  $N$ . Here  $\text{Pf}(\lambda)$  is the Pfaffian of the antisymmetric bilinear form on  $\mathfrak{n}/\mathfrak{z}$  given by  $b_\lambda(x, y) = \lambda([x, y])$ . The square integrable representations of  $N$  are certain easily-constructed representations  $\pi_\lambda$  where  $\lambda \in \mathfrak{z}^*$  with  $\text{Pf}(\lambda) \neq 0$ , Plancherel almost irreducible unitary representations of  $N$  are square integrable, and up to an explicit constant  $|\text{Pf}(\lambda)|$  is the Plancherel density of the unitary dual  $\hat{N}$  at  $\pi_\lambda$ .

This theory has proved to have serious analytic consequences [12].

In this paper we present an extension of that theory. Under certain conditions, the nilpotent Lie group  $N$  has a particular decomposition into subgroups that have square integrable representations, and the Plancherel formula then is synthesized explicitly in terms of the Plancherel formulae of those subgroups. Many of our calculations are on the Lie algebra level, reflecting the setting of the square integrability conditions mentioned above.

The guiding example for these decompositions is that of the upper triangular real matrices. We go through it separately in Section 2 for the decompositions, Section 3 for the square integrability, and Section 4 for the Plancherel formula. We separate out the upper triangular groups for two reasons. First, they give a clear illustration of the technical conditions that we need more generally to decompose the group and to reconstitute the Plancherel formula. Second, and equally important, they appear in many situations and are of independent interest. The precise conditions are given in (5.1).

The decompositions for upper triangular matrices were suggested by the work of María L. Barberis and Isabel G. Dotti on abelian complex structures; see [2], [3] and [5]. I thank them for discussions of their work in progress [4] on those structures.

The general formulation and the resulting Plancherel formulae are the content of Section 5. There we extend the material of Sections 2, 3 and 4 to a much more general setting.

In Section 6 we verify the conditions of Section 5 for the unipotent radicals of parabolic subgroups of real semisimple Lie groups. This has many potential applications in differential geometry, in hypoelliptic differential equations, and in harmonic analysis. For example, in the case of cuspidal parabolics, it has the potential of simplifying some of the integrations in Harish–Chandra’s theory of the constant term. These unipotent radical examples also appear in many other geometric and analytic settings.

Finally, in Section 7 we consider the case where our connected simply connected nilpotent Lie group  $N$  has a discrete co-compact subgroup  $\Gamma$  that fits into the pattern of Section 5. We show that the compact nilmanifold  $N/\Gamma$  has a corresponding foliation and derive analytic results analogous to those of Theorem 5.3. These results include multiplicity formulae for stepwise square integrable representations as summands of the regular representation  $\text{Ind}_\Gamma^N(1_\Gamma)$  of  $N$  on  $L^2(N/\Gamma)$ . They apply in particular to the nilradicals of minimal parabolic subgroups, as studied in Section 6.

Section 8 is an appendix demonstrating the root structure of the key technical tool, Lemma 6.8, behind Theorem 6.16. I expect that it will only appear in the arXiv version of this paper.

## 2 Decomposition of Upper Triangular Matrices

Let  $\mathfrak{n}$  denote the real Lie algebra of  $\ell \times \ell$  matrices with zeroes on and below the diagonal, and let  $N$  the corresponding unipotent Lie group of  $\ell \times \ell$  matrices with zeroes below the diagonal and ones on the diagonal.

As usual  $e_{i,j} \in \mathfrak{n}$  denotes the matrix with 1 in row  $i$  column  $j$  and zeroes elsewhere, so  $\mathfrak{n}$  is the span of  $\{e_{i,j} \mid 1 \leq i < j \leq \ell\}$ . Here  $[e_{i,j}, e_{m,n}]$  is  $e_{i,n}$  if  $i < j = m < n$ ,  $-e_{m,j}$  if  $m < n = i < j$ , 0 in all other cases. Thus, for  $1 \leq r \leq \lfloor \frac{\ell}{2} \rfloor$ , we define

$$(2.1) \quad \begin{aligned} \mathfrak{m}_r &:= \text{Span}\{e_{r,s} \mid r+1 \leq s \leq \ell-r\} \cup \text{Span}\{e_{q,\ell-r+1} \mid r+1 \leq q \leq \ell-r\} \cup e_{r,\ell-r+1}\mathbb{R} \text{ and} \\ \mathfrak{n}_r &:= \mathfrak{m}_1 + \mathfrak{m}_2 + \cdots + \mathfrak{m}_r = \text{Span}\{e_{i,j} \mid i \leq r \leq \ell-r+1 < j\}. \end{aligned}$$

Then  $\mathfrak{m}_r$  is a subalgebra of  $\mathfrak{n}$  that is isomorphic to the Heisenberg algebra  $\mathfrak{h}_{\ell-2r}$  of dimension  $2(\ell-2r)+1$ ; it has center  $\mathfrak{z}_r := e_{r,\ell-r+1}\mathbb{R}$ . Note that

$$(2.2) \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{n}_{\min(i,j)}, \text{ so each } \mathfrak{n}_r \text{ is an ideal in } \mathfrak{n}.$$

In particular we have semidirect sum decompositions

$$(2.3) \quad \mathfrak{n}_r = \mathfrak{n}_{r-1} \ltimes \mathfrak{m}_r$$

and a filtration

$$(2.4) \quad \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \cdots \subset \mathfrak{n}_{\lfloor \ell/2 \rfloor} = \mathfrak{n}$$

by ideals.

Fix the positive definite inner product on  $\mathfrak{n}$  in which the  $e_{i,j}$  are orthonormal. Define  $J_r : \mathfrak{n} \rightarrow \mathfrak{n}$  by  $\langle J_r(x), y \rangle = \langle e_{r,\ell-r+1}, [x, y] \rangle$ . Its image is the non-central part

$$(2.5) \quad \text{Span}\{e_{r,s} \mid r+1 \leq s \leq \ell-r\} \cup \text{Span}\{e_{q,\ell-r+1} \mid r+1 \leq q \leq \ell-r\} = \mathfrak{m}_r \cap \mathfrak{v}_r$$

of  $\mathfrak{m}_r$  and its kernel is the central part  $\mathfrak{z}_r = e_{r,\ell-r+1}\mathbb{R}$ . That is the connection with abelian complex structures mentioned in the Introduction.

Group structure here follows algebra structure immediately, as all the groups are unipotent and thus equal to the exponential image of their Lie algebras. Thus we have closed connected subgroups  $M_r \cong H_{\ell-2r}$  and normal closed connected subgroups  $N_r = M_1 M_2 \cdots M_r$  in  $N$ , and semidirect product decompositions  $N_r = N_{r-1} \rtimes M_r$ .

### 3 Square Integrability for Upper Triangular Matrices

Now we cascade down antidiagonally from the upper right hand corner. For convenience let  $m := \lfloor \frac{\ell}{2} \rfloor$ . If  $r < m$  then  $M_r$  is the Heisenberg group of dimension  $2(\ell - 2r) + 1$ . If  $\ell$  is odd then  $M_m$  is the 3-dimensional Heisenberg group, and if  $\ell$  is even then  $M_m \cong \mathbb{R}$  is the 1-dimensional vector group. The point is that Plancherel-almost-every irreducible unitary representation of  $M_r$  is a representation  $\pi_{\lambda_r}$  specified by a nonzero linear functional  $\lambda_r$  on the center  $\mathfrak{z}_r$  of  $\mathfrak{m}_r$ , and that representation has matrix coefficients in  $L^2(M_r/Z_r)$ . Write  $\mathcal{H}_{\pi_{\lambda_r}}$  for the representation space, or just  $\mathcal{H}_r$  if there is no chance of confusion.

Let  $\lambda = \lambda_1 + \dots + \lambda_m$  where  $0 \neq \lambda_r \in \mathfrak{z}_r$  for  $1 \leq r \leq m$ . We are going to put together the square integrable representations  $\pi_{\lambda_r} \in \widehat{M}_r$  to form a representation  $\pi_r \in \widehat{N}$ . This will be a recursion on  $r$  and we will need the

$$S_r = Z_1 Z_2 \dots Z_r = S_{r-1} \times Z_r$$

for that recursive construction.

**Lemma 3.1.**  $M_r$  centralizes  $S_{r-1}$ .

*Proof.* The Lie algebra  $\mathfrak{s}_{r-1}$  is spanned by the  $e_{j, \ell+1-j}$  for  $j = 1, \dots, r-1$ . Let  $e_{u,v} \in \mathfrak{m}_r$ . Then  $[e_{u,v}, e_{j, \ell+1-j}] = 0$  because  $v > j$  and  $u < \ell + 1 - j$ .  $\square$

Express  $N_2$  as the semidirect product  $N_1 \rtimes M_2$ . Plancherel-almost-every irreducible unitary representation of  $N_1 = M_1$  is a representation  $\pi_{\lambda_1}$  specified by a nonzero linear functional  $\lambda_1 \in \mathfrak{z}_1^*$ . View  $\lambda_1$  as an element of  $\mathfrak{n}_1^*$  that vanishes on the non-central matrices  $e_{i,j}$  in  $\mathfrak{n}_1$ . Choose an invariant polarization  $\mathfrak{p}'_1 \subset \mathfrak{n}_2$  for the linear functional  $\lambda'_1 \in \mathfrak{n}_2^*$  that agrees with  $\lambda_1$  on  $\mathfrak{n}_1$  and vanishes on  $\mathfrak{m}_2$ . Lemma 3.1 implies  $\text{ad}^*(\mathfrak{m}_2)(\lambda'_1)|_{\mathfrak{z}_1 + \mathfrak{m}_2} = 0$ , so  $\mathfrak{p}'_1 = \mathfrak{p}_1 + \mathfrak{m}_2$  where  $\mathfrak{p}_1$  is an invariant polarization for the linear functional  $\lambda_1 \in \mathfrak{n}_1^*$ . The associated representations are  $\pi_{\lambda'_1} \in \widehat{N}_2$  and  $\pi_{\lambda_1} \in \widehat{N}_1$ . Note that  $N_2/P'_1 = N_1/P_1$ , so the representation spaces  $\mathcal{H}_{\pi_{\lambda'_1}} = L^2(N_2/P'_1) = L^2(N_1/P_1) = \mathcal{H}_{\pi_{\lambda_1}}$ . In other words,  $\pi_{\lambda'_1}$  extends  $\pi_{\lambda_1}$  to a unitary representation of  $N_2$  on the same Hilbert space  $\mathcal{H}_{\pi_{\lambda_1}}$ , and  $d\pi_{\lambda'_1}(\mathfrak{z}_2) = 0$ . Now the Mackey Little Group method gives us

**Lemma 3.2.** *The irreducible unitary representations of  $N_2$ , whose restrictions to  $N_1$  are multiples of  $\pi_{\lambda_1}$ , are the  $\pi_{\lambda'_1} \widehat{\otimes} \gamma$  where  $\gamma \in \widehat{M}_2 = \widehat{N}_2/\widehat{N}_1$ .*

Given nonzero  $\lambda_1 \in \mathfrak{z}_1^*$  and  $\lambda_2 \in \mathfrak{z}_2^*$  we have representations  $\pi_{\lambda_1} \in \widehat{M}_1$  and  $\pi_{\lambda_2} \in \widehat{M}_2$  with coefficients in  $L^2(M_1/Z_1)$  and  $L^2(M_2/Z_2)$  respectively. Using the notation of Lemma 3.2 we define

$$(3.3) \quad \pi_{\lambda_1 + \lambda_2} \in \widehat{N}_2 \text{ by } \pi_{\lambda_1 + \lambda_2} = \pi'_{\lambda_1} \widehat{\otimes} \pi_{\lambda_2}.$$

We now use the square integrability of  $\pi_{\lambda_1}$  and  $\pi_{\lambda_2}$  for some square integrability of  $\pi_{\lambda_1 + \lambda_2}$ .

**Proposition 3.4.** *The coefficients  $f_{z,w}(xy) = \langle z, \pi_{\lambda_1 + \lambda_2}(xy)w \rangle$  of  $\pi_{\lambda_1 + \lambda_2}$  are in  $L^2(N_2/S_2)$ , in fact satisfy  $\|f_{z,w}\|_{L^2(N_2/S_2)}^2 = \frac{\|z\|^2 \|w\|^2}{\deg(\pi_{\lambda_1}) \dots \deg(\pi_{\lambda_r})}$ .*

*Proof.* We write  $\mathcal{H}_r$  for the representation space of  $\pi_{\lambda_r}$ .  $\mathcal{H}_1$  also is the representation space for  $\pi_{\lambda'_1}$ , so  $\pi_{\lambda_1 + \lambda_2}$  has representation space  $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ . Choose nonzero vectors  $u, v \in \mathcal{H}_1$  and  $u', v' \in \mathcal{H}_2$ . We need only prove that the function  $f(x, y) = \langle u, \pi'_{\lambda_1}(xy)v \rangle \langle u', \pi_{\lambda_2}(y)v' \rangle$ ,  $x \in M_1$  and  $y \in M_2$ , satisfies  $\|f\|_{L^2(N_2/S_2)}^2 = \left( \frac{\|u\|^2 \|v\|^2}{\deg(\pi_{\lambda_1})} \right) \left( \frac{\|u'\|^2 \|v'\|^2}{\deg(\pi_{\lambda_2})} \right)$ , so that the coefficients  $xy \mapsto \langle u \otimes u', \pi_{\lambda_1 + \lambda_2}(xy)(v \otimes v') \rangle$  of decomposable vectors are in  $L^2(N_2/S_2)$ . For that, let  $\{z_i\}$  and  $\{w_j\}$  be complete orthonormal sets in  $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ . Suppose that both  $\sum |a_{i,j}|^2$  and  $\sum |b_{i,j}|^2$  are finite, so  $z = \sum a_{i,j} z_i \otimes w_j$  and  $w = \sum b_{i,j} z_i \otimes w_j$  are general elements of  $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ . Then the coefficient  $\langle x, \pi_{\lambda_1 + \lambda_2}(xy)w \rangle = \sum a_{i,j} \overline{b_{i',j'}} \langle z_i, \pi_{\lambda_1 + \lambda_2}(xy)w_j \rangle$  has square  $L^2(N_2/S_2)$ -norm  $\frac{1}{\deg(\pi_{\lambda_1})} \frac{1}{\deg(\pi_{\lambda_2})} \sum |a_{i,j}|^2 \sum |b_{i',j'}|^2 = \frac{\|z\|^2 \|w\|^2}{\deg(\pi_{\lambda_1}) \deg(\pi_{\lambda_2})} < \infty$ .

In order to integrate  $|f|^2$  over  $N_2 = M_1 M_2$  modulo  $S_2 = Z_1 Z_2$  we use the fact that the action of  $M_2$  on  $\mathfrak{m}_1$  is unipotent, so there is a measure preserving decomposition

$$(3.5) \quad N_2/S_2 = (M_1/Z_1) \times (N_2/Z_2).$$

Using the extension of Schur Orthogonality to representations with coefficients that are square integrable modulo the center of the group, and writing  $v_y$  for  $\pi_{\lambda'_1}(y)v$ , we compute

$$\begin{aligned}
\|f\|_{L^2(N_2/S_2)}^2 &= \int_{N_2/S_2} |\langle u, \pi'_{\lambda'_1}(xy)v \rangle|^2 |\langle u', \pi_{\lambda_2}(y)v' \rangle|^2 d(xyZ_1Z_2) \\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v' \rangle|^2 \left( \int_{M_1/Z_1} |\langle u, \pi_{\lambda'_1}(xy)v \rangle|^2 d(xZ_1) \right) d(yZ_2) \\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v' \rangle|^2 \left( \int_{M_1/Z_1} |\langle u, \pi_{\lambda'_1}(x)v_y \rangle|^2 d(xZ_1) \right) d(yZ_2) \\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v' \rangle|^2 \left( \int_{M_1/Z_1} |\langle u, \pi_{\lambda_1}(x)v_y \rangle|^2 d(xZ_1) \right) d(yZ_2) \\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v' \rangle|^2 \frac{\|u\|^2 \|v_y\|^2}{\deg(\pi_{\lambda_1})} d(yZ_2) \\
&= \left( \frac{\|u\|^2 \|v_y\|^2}{\deg(\pi_{\lambda_1})} \right) \left( \frac{\|u'\|^2 \|v'\|^2}{\deg(\pi_{\lambda_2})} \right) = \left( \frac{\|u\|^2 \|v\|^2}{\deg(\pi_{\lambda_1})} \right) \left( \frac{\|u'\|^2 \|v'\|^2}{\deg(\pi_{\lambda_2})} \right) \\
&= \frac{\|u \otimes u'\|^2 \|v \otimes v'\|^2}{\deg(\pi_{\lambda_1}) \deg(\pi_{\lambda_2})} < \infty.
\end{aligned}$$

That completes the proof of Proposition 3.4.  $\square$

Proposition 3.4 starts our recursive construction. More generally,  $N_{r+1}$  is the semidirect product  $N_r \rtimes M_{r+1}$ . We fix nonzero  $\lambda_i \in \mathfrak{z}_i^*$  for  $1 \leq i \leq r+1$ , and we start with the representation  $\pi_{\lambda_1+\dots+\lambda_r}$  constructed step by step from the square integrable representations  $\pi_{\lambda_i} \in \widehat{M_i}$  for  $1 \leq i \leq r$ . The representation space  $\mathcal{H}_{\pi_{\lambda_1+\dots+\lambda_r}} = \mathcal{H}_{\pi_{\lambda_1}} \widehat{\otimes} \dots \mathcal{H}_{\pi_{\lambda_r}}$ . The coefficients of  $\pi_{\lambda_1+\dots+\lambda_r}$  have absolute value in  $L^2(N_r/S_r)$ . In fact they satisfy

$$\|f_{z,w}\|_{L^2(N_r/S_r)}^2 = \frac{\|z\|^2 \|w\|^2}{\deg(\pi_{\lambda_1}) \dots \deg(\pi_{\lambda_r})}.$$

Then  $\pi_{\lambda_1+\dots+\lambda_r}$  extends to a representation  $\pi'_{\lambda_1+\dots+\lambda_r}$  of  $L_{r+1}$  on the same Hilbert space  $\mathcal{H}_{\pi_{\lambda_1+\dots+\lambda_r}}$ , and it satisfies  $d\pi'_{\lambda_1+\dots+\lambda_r}(\mathfrak{z}_{r+1}) = 0$ . Thus, as in Lemma 3.2,

**Lemma 3.6.** *The irreducible unitary representations of  $N_{r+1}$ , whose restrictions to  $N_r$  are multiples of  $\pi_{\lambda_1+\dots+\lambda_r}$ , are the  $\pi'_{\lambda_1+\dots+\lambda_r} \widehat{\otimes} \gamma$  where  $\gamma \in \widehat{M_{r+1}} = \widehat{N_{r+1}/N_r}$ .*

Recall  $0 \neq \lambda_{r+1} \in \mathfrak{z}_{r+1}^*$  and the square integrable representation  $\pi_{\lambda_{r+1}}$  of  $M_{r+1} = L_{r+1}/L_r$ . Computing exactly as in Proposition 3.4, we define  $\pi_{\lambda_1+\dots+\lambda_{r+1}} = \pi'_{\lambda_1+\dots+\lambda_r} \widehat{\otimes} \pi_{\lambda_{r+1}}$  and conclude that

**Proposition 3.7.** *The coefficients  $f_{z,w}(x_1 \dots x_{r+1}) = \langle z, \pi_{\lambda_1+\dots+\lambda_{r+1}}(x_1 x_2 \dots x_{r+1})w \rangle$  of  $\pi_{\lambda_1+\dots+\lambda_{r+1}}$  are in  $L^2(N_{r+1}/S_{r+1})$ , in fact satisfy  $\|f_{z,w}\|_{L^2(N_{r+1}/S_{r+1})}^2 = \frac{\|z\|^2 \|w\|^2}{\deg(\pi_{\lambda_1}) \dots \deg(\pi_{\lambda_{r+1}})}$ .*

Since the  $M_r$  are Heisenberg groups, except that  $M_m$ , the last one, is 1-dimensional abelian in case the size  $\ell$  of the matrices is even, we have  $\deg \pi_{\lambda_r} = |\lambda_r|^{d_r}$  where  $\dim M_r = 2d_r + 1$  and  $d_r = \ell - 2r$ . Proposition 3.7 is the recursion step for our construction, and the end case  $r+1 = m$  is

**Theorem 3.8.** *Let  $0 \neq \lambda_r \in \mathfrak{z}_r^*$  for  $1 \leq r \leq m$  and set  $\lambda = \lambda_1 + \dots + \lambda_m$ . Denote  $\deg(\pi_\lambda) = \deg(\pi_{\lambda_1}) \dots \deg(\pi_{\lambda_m})$ . Then the coefficients  $f_{z,w}(x) = \langle z, \pi_\lambda(x)w \rangle$  of the irreducible unitary representation  $\pi_\lambda$  on  $N$  are in  $L^2(N/S)$  and satisfy  $\|f_{z,w}\|_{L^2(N/S)}^2 = \frac{\|z\|^2 \|w\|^2}{\deg(\pi_\lambda)} = \|z\|^2 \|w\|^2 / \prod |\lambda_r|^{\ell-2r}$ .*

**Definition 3.9.** The representations  $\pi_\lambda$ , constructed as just above, are the *stepwise square integrable* representations of  $N$  relative to the decompositions (2.1), (2.3) and (2.4).  $\diamond$

## 4 Plancherel Formula for Upper Triangular Matrices

The Plancherel measure for the group  $M_r$  is  $2^{d_r} d_r! |\lambda_r|^{d_r} d\lambda_r$  where  $\dim M_r = 2d_r + 1$  and  $d\lambda_r$  is Lebesgue measure on  $\mathfrak{z}_r^*$ . If  $f \in L^1(M_r)$  we have  $\hat{\pi}_{\lambda_r}(f) = \int_{M_r} f(x_r) \pi_{\lambda_r}(x_r) dx_r$ . One version of the Plancherel formula for  $M_r$  is

$$(4.1) \quad \|f\|_{L^2(M_r)}^2 = 2^{d_r} d_r! \int_{\mathfrak{z}_r^*} \|\hat{\pi}_{\lambda_r}(f)\|_{HS}^2 |\lambda_r|^{d_r} d\lambda_r \quad (d_r = \ell - 2r)$$

where  $f \in L^1(M_r) \cap L^2(M_r)$  and  $\|\cdot\|_{HS}$  is Hilbert–Schmidt norm, and another is

$$(4.2) \quad f(x) = c_r \int_{\mathfrak{z}_r^*} \Theta_{\pi_{\lambda_r}}(r_x f) |\lambda_r|^{d_r} d\lambda_r$$

where  $\Theta_{\pi_{\lambda_r}}$  is the distribution character of  $\pi_{\lambda_r}$ , given by  $\Theta_{\pi_{\lambda_r}}(h) = \text{trace } \pi_{\lambda_r}(h)$ ,  $f \in C_c^\infty(M_r)$ ,  $c_r = 2^{d_r} d_r!$ , and  $r_x$  is right translation of functions,  $(r_x f)(g) = f(gx)$ . As we will see in a moment from the formula, the distribution  $\Theta_{\pi_{\lambda_r}}$  is tempered, i.e. extends by continuity to the Schwartz space  $\mathcal{C}(M_r)$ .

To make this explicit one needs the character formula for  $\pi_{\lambda_r}$ , i.e. the formula for the tempered distribution  $\Theta_{\pi_{\lambda_r}}$ . That is given as follows. Define  $h_1 \in C_c^\infty(\mathfrak{m}_r)$  by  $h_1(\xi) = h(\exp(\xi))$ . The geometric tangent space of  $\text{Ad}^*(M_r)\lambda_r$  is the coadjoint orbit  $\text{Ad}^*(M_r)\lambda_r$  itself, the affine hyperplane  $\lambda_r + \mathfrak{z}_r^\perp$  in  $\mathfrak{m}_r^* \cong (\mathfrak{m}_r/\mathfrak{z}_r)^*$ . We use Lebesgue measure  $d\nu_r$  on  $(\mathfrak{m}_r/\mathfrak{z}_r)^*$  normalized so that Fourier transform is an isometry from  $L^2(\mathfrak{m}_r/\mathfrak{z}_r)$  onto  $L^2(\mathfrak{m}_r/\mathfrak{z}_r)^*$ , and we translate  $d\nu_r$  to a measure  $d\nu_{\lambda_r}$  on the orbit. Then, from [9] and [8],

$$(4.3) \quad \Theta_{\pi_{\lambda_r}}(h) = c_r^{-1} |\lambda_r|^{-d_r} \int_{\text{Ad}^*(M_r)\lambda_r} \widehat{h_1}(\xi) d\nu_{\lambda_r}(\xi) \quad (d_r = \ell - 2r)$$

where  $\widehat{h_1}$  is the Fourier transform of  $h_1$ . For all this see [8, Theorem 6 and its proof].

In order to extend these results from one group  $M_r$  to the upper triangular group  $N$  we first need

**Proposition 4.4.** *Plancherel measure on  $\widehat{N}$  is concentrated on  $\{\pi_\lambda \mid \lambda = \lambda_1 + \dots + \lambda_m, 0 \neq \lambda_r \in \mathfrak{z}_r^* \ \forall r\}$ .*

*Proof.* If  $\zeta \in \mathfrak{s}^*$  then  $e^{2\pi\sqrt{-1}\zeta} : \xi \mapsto e^{2\pi\sqrt{-1}\zeta(\log \xi)}$  on  $S$  is a unitary character on  $S$ . Denote the induced representation  $\tilde{\zeta} = \text{Ind}_S^N(e^{2\pi\sqrt{-1}\zeta})$ . Induction by stages says that the left regular representation of  $N$  is  $\text{Ind}_{\{1\}}^N(1) = \int_{\mathfrak{s}^*} \tilde{\zeta} d\zeta$  where  $d\zeta$  is Lebesgue measure on  $\mathfrak{s}^*$ . Let

$$(4.5) \quad \mathfrak{t}^* = \{\lambda = \lambda_1 + \dots + \lambda_m, 0 \neq \lambda_r \in \mathfrak{z}_r^* \ \forall r\} \text{ and } P(\lambda) = \lambda_1^{d_1} \lambda_2^{d_2} \dots \lambda_m^{d_m}.$$

Since  $P$  is not identically zero we can ignore its zero set in the direct integral, so left regular representation of  $N$  is  $\text{Ind}_{\{1\}}^N(1) = \int_{\mathfrak{t}^*} \tilde{\zeta} d\zeta$ .  $\square$

Next, we break up the bilinear form  $b_\lambda$ .

**Lemma 4.6.** *Decompose each  $\mathfrak{m}_r = \mathfrak{z}_r + \mathfrak{v}_r$  where  $\mathfrak{v}_r$  is the span of the  $e_{i,j}$  in  $\mathfrak{m}_r$  but not in  $\mathfrak{z}_r$ , and similarly  $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$ . If  $\lambda \in \mathfrak{t}^*$  then the antisymmetric bilinear form  $b_\lambda$  on  $\mathfrak{v}$  is the direct sum  $b_{\lambda_1} \oplus \dots \oplus b_{\lambda_m}$  of nondegenerate bilinear forms on the  $\mathfrak{v}_r$ . Equivalently, if  $r \neq t$  then  $[\mathfrak{m}_r, \mathfrak{m}_t] \subset \mathfrak{v}$ .*

*Proof.* The equivalence is clear from the definition of  $b_\lambda$ . Let  $\rho$  denote reflection on the antidiagonal. Suppose that  $e_{i,j} \in \mathfrak{m}_r$  and  $e_{a,b} \in \mathfrak{m}_t$  with  $r \neq t$ . Then  $e_{i,j}e_{a,b}$  is on the antidiagonal if and only if  $\rho(e_{i,j}) = e_{a,b}$ , and this happens if and only if  $e_{a,b}e_{i,j}$  is on the antidiagonal. As  $\rho(\mathfrak{m}_r) = \mathfrak{m}_r$  and  $\rho(\mathfrak{m}_t) = \mathfrak{m}_t$ , it follows that  $[\mathfrak{m}_r, \mathfrak{m}_t] \subset \mathfrak{v}$ .  $\square$

Given  $\lambda \in \mathfrak{t}^*$ , the coadjoint orbit  $\mathcal{O}(\lambda) := \text{Ad}^*(N)\lambda$  is just  $\text{Ad}(M_1)\lambda_1 \times \dots \times \text{Ad}(M_m)\lambda_m$ , and we have the measure  $d\nu_\lambda = d\nu_{\lambda_1} \times \dots \times d\nu_{\lambda_m}$  on it. Denote  $c = c_1 c_2 \dots c_m = 2^{d_1 + \dots + d_m} d_1! d_2! \dots d_m!$  where we recall  $d_r = \frac{1}{2}(\dim \mathfrak{m}_r - 1) = \ell - 2r$ . Let  $f \in C_c^\infty(N)$  (or more generally  $f \in \mathcal{C}(N)$ ) and  $\widehat{f_1}$  the classical Fourier transform of the lift  $f_1(\xi) = f(\exp(\xi))$  of  $f$  to  $\mathfrak{n}$ . We use Lebesgue measure  $d\nu$  on  $(\mathfrak{m}/\mathfrak{s})^*$  normalized so that Fourier transform is an isometry of  $L^2(\mathfrak{m}/\mathfrak{s})$  onto  $L^2(\mathfrak{m}/\mathfrak{s})^*$ .

Now, exactly as in (4.2) and (4.3) we combine the result [9, Theorem, p. 17] of Pukánszky with the method of [8, proof of Theorem 6] to obtain

**Theorem 4.7.** *Let  $N$  be the group of real strictly triangular  $\ell \times \ell$  matrices,  $\mathfrak{m}_r$  and  $\mathfrak{n}_r$  the algebras of Section 2, and  $M_r$  and  $N_r$  the corresponding analytic subgroups of  $N$ . Let  $\lambda = \lambda_1 + \dots + \lambda_m \in \mathfrak{t}^*$ , and  $P(\lambda) = \lambda_1^{\ell-2} \lambda_2^{\ell-4} \dots \lambda_m^{\ell-2m}$ , as in (4.5). Then  $\pi_\lambda \in \widehat{N}$  has distribution character*

$$\Theta_{\pi_\lambda}(f) = \text{trace } \dot{\pi}_\lambda(f) = \frac{1}{c} \frac{1}{|P(\lambda)|} \int_{\mathcal{O}(\lambda)} \widehat{f}_1(\xi) d\nu(\xi)$$

and  $N$  has Plancherel formula

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f) |P(\lambda)| d\lambda.$$

## 5 General Theory

Here's what we need to extend our considerations beyond the group of upper triangular matrices. The connected simply connected nilpotent Lie group should decompose as

$$(5.1) \quad \begin{aligned} N &= M_1 M_2 \dots M_{m-1} M_m \text{ where} \\ (a) & \text{ each factor } M_r \text{ has unitary representations with coefficients in } L^2(M_r/Z_r), \\ (b) & \text{ each } N_r := M_1 M_2 \dots M_r \text{ is a normal subgroup of } N \text{ with } N_r = N_{r-1} \rtimes M_r \text{ semidirect,} \\ (c) & \text{ decompose } \mathfrak{m}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n} = \mathfrak{s} + \mathfrak{v} \text{ as vector direct sums where} \\ & \mathfrak{s} = \bigoplus \mathfrak{z}_r \text{ and } \mathfrak{v} = \bigoplus \mathfrak{v}_r; \text{ then } [\mathfrak{m}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{v} \text{ for } r > s. \end{aligned}$$

In order to follow the arguments leading to Theorem 4.7, we denote

$$(5.2) \quad \begin{aligned} (a) & d_r = \frac{1}{2} \dim(\mathfrak{m}_r/\mathfrak{z}_r) \text{ so } \frac{1}{2} \dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \dots + d_m, \text{ and } c = 2^{d_1 + \dots + d_m} d_1! d_2! \dots d_m! \\ (b) & b_{\lambda_r} : (x, y) \mapsto \lambda([x, y]) \text{ viewed as a bilinear form on } \mathfrak{m}_r/\mathfrak{z}_r \\ (c) & S = Z_1 Z_2 \dots Z_m = Z_1 \times \dots \times Z_m \text{ where } Z_r \text{ is the center of } M_r \\ (d) & P : \text{ polynomial } P(\lambda) = \text{Pf}(b_{\lambda_1}) \text{Pf}(b_{\lambda_2}) \dots \text{Pf}(b_{\lambda_m}) \text{ on } \mathfrak{s}^* \\ (e) & \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\} \\ (f) & \pi_\lambda \in \widehat{N} \text{ where } \lambda \in \mathfrak{t}^* : \text{ irreducible unitary rep. of } N = M_1 M_2 \dots M_m \text{ as in Section 4} \end{aligned}$$

Proposition 4.4 extends immediately to this setting: Plancherel measure is concentrated on  $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$ . It is slightly more delicate to extend Lemma 4.6, but (5.1)(c) does the job.

**Theorem 5.3.** *Let  $N$  be a connected simply connected nilpotent Lie group that satisfies (5.1). Then Plancherel measure for  $N$  is concentrated on  $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$ . If  $\lambda \in \mathfrak{t}^*$ , and if  $u$  and  $v$  belong to the representation space  $\mathcal{H}_{\pi_\lambda}$  of  $\pi_\lambda$ , then the coefficient  $f_{u,v}(x) = \langle u, \pi_\nu(x)v \rangle$  satisfies*

$$(5.4) \quad \|f_{u,v}\|_{L^2(N/S)}^2 = \frac{\|u\|^2 \|v\|^2}{|P(\lambda)|}.$$

The distribution character  $\Theta_{\pi_\lambda}$  of  $\pi_\lambda$  satisfies

$$(5.5) \quad \Theta_{\pi_\lambda}(f) = c^{-1} |P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f}_1(\xi) d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N)$$

where  $\mathcal{C}(N)$  is the Schwartz space,  $f_1$  is the lift  $f_1(\xi) = f(\exp(\xi))$ ,  $\widehat{f}_1$  is its classical Fourier transform,  $\mathcal{O}(\lambda)$  is the coadjoint orbit  $\text{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$ , and  $d\nu_\lambda$  is the translate of normalized Lebesgue measure from  $\mathfrak{v}^*$  to  $\text{Ad}^*(N)\lambda$ . The Plancherel formula on  $N$  is

$$(5.6) \quad f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f) |P(\lambda)| d\lambda \text{ for } f \in \mathcal{C}(N).$$

**Definition 5.7.** The representations  $\pi_\lambda$  of (5.2(f)) are the *stepwise square integrable* representations of  $N$  relative to (5.1).  $\diamond$

## 6 Iwasawa Decompositions

Let  $G$  be a real reductive Lie group. We now carry out the program of Section 5 for the groups  $N$  of Iwasawa decompositions  $G = KAN$ . Let  $m = \text{rank}_{\mathbb{R}} G = \dim_{\mathbb{R}} A$  and notice that we've done the case  $G = SL(m+1; \mathbb{R})$ . The idea is to use the Kostant cascade construction of strongly orthogonal roots:  $\beta_1$  is the maximal root,  $\beta_{r+1}$  is a maximum among the positive roots orthogonal to  $\{\beta_1, \dots, \beta_r\}$ , etc.

We fix an Iwasawa decomposition  $G = KAN$ . As usual, write  $\mathfrak{k}$  for the Lie algebra of  $K$ ,  $\mathfrak{a}$  for the Lie algebra of  $A$ , and  $\mathfrak{n}$  for the Lie algebra of  $N$ . Complete  $\mathfrak{a}$  to a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Then  $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$  with  $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ . Now we have root systems

- $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ : roots of  $\mathfrak{g}_{\mathbb{C}}$  relative to  $\mathfrak{h}_{\mathbb{C}}$  (ordinary roots),
- $\Delta(\mathfrak{g}, \mathfrak{a})$ : roots of  $\mathfrak{g}$  relative to  $\mathfrak{a}$  (restricted roots),
- $\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\}$  (nonmultipliable restricted roots).

Here  $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \neq 0\}$ . Further,  $\Delta(\mathfrak{g}, \mathfrak{a})$  and  $\Delta_0(\mathfrak{g}, \mathfrak{a})$  are root systems in the usual sense. Any positive root system  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  defines positive systems

- $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \neq 0\}$  and  $\Delta_0^+(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a}) \cap \Delta^+(\mathfrak{g}, \mathfrak{a})$ .

We can (and do) choose  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  so that

- $\mathfrak{n}$  is the sum of the positive restricted root spaces and
- if  $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  and  $\gamma|_{\mathfrak{a}} \in \Delta^+(\mathfrak{g}, \mathfrak{a})$  then  $\gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ .

Two roots are called *strongly orthogonal* if their sum and their difference are not roots. Then they are orthogonal. We define

$$(6.1) \quad \begin{aligned} \beta_1 &\in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and} \\ \beta_{r+1} &\in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ orthogonal to all } \beta_i \text{ with } i \leq r \end{aligned}$$

Then the  $\beta_r$  are mutually strongly orthogonal. This is Kostant's cascade construction. Note that each  $\beta_r \in \Delta_0^+(\mathfrak{g}, \mathfrak{a})$ . Also note that  $\beta_1$  is unique if and only if  $\Delta(\mathfrak{g}, \mathfrak{a})$  is irreducible.

For  $1 \leq r \leq m$  define

$$(6.2) \quad \begin{aligned} \Delta_1^+ &= \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and} \\ \Delta_{r+1}^+ &= \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \dots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}. \end{aligned}$$

**Lemma 6.3.** *If  $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$  then either  $\alpha \in \{\beta_1, \dots, \beta_m\}$  or  $\alpha$  belongs to exactly one of the sets  $\Delta_r^+$ .*

*Proof.* Suppose that  $\alpha \notin \{\beta_1, \dots, \beta_m\}$  and that  $\alpha \notin \Delta_r^+$  for any  $r$ . As  $\alpha \notin \Delta_1^+$  it is strongly orthogonal to  $\beta_1$ . Then as  $\alpha \notin \Delta_2^+$  it is strongly orthogonal to  $\beta_2$  as well. Continuing,  $\alpha$  is strongly orthogonal to each of the  $\beta_r$ , contradicting maximality of  $\{\beta_1, \dots, \beta_m\}$ .  $\square$

**Lemma 6.4.** *The set  $\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i \text{ for } i < r \text{ and } \langle \alpha, \beta_r \rangle > 0\}$ . In particular,  $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{m}_t$  where  $t = \min\{r, s\}$ .*

*Proof.* Let  $\alpha \in \Delta^+$  such that (i)  $\alpha \perp \beta_i$  for  $i < r$  and (ii)  $\langle \alpha, \beta_r \rangle > 0$ . Here (ii) shows that  $\beta_r - \alpha$  is a root. If it's negative then  $\alpha > \beta_r$ , contradicting maximality of  $\beta_r$  for the property of being orthogonal to  $\beta_i$  for every  $i < r$ . So  $\beta_r - \alpha \in \Delta^+$ . Let  $s$  be the smallest integer such that  $\alpha \in \Delta_s^+$ . This relies on Lemma 6.3. As argued a moment ago,  $\beta_s + \alpha$  is not a root. If  $s < r$  then (i) says that  $\alpha$  is strongly orthogonal to  $\beta_s$ , contradicting  $\alpha \in \Delta_s^+$ . Thus  $r = s$  and  $\alpha \in \Delta_r^+$ .

Conversely we want to show that  $\alpha \in \Delta_r^+$  implies  $\alpha \perp \beta_i$  for  $i < r$  and  $\langle \alpha, \beta_r \rangle > 0$ . This is clear for  $r = 1$ . We assume it for  $r < t$ , for a fixed  $t \leq m$ , and prove it for  $r = t$ . Let  $\alpha \in \Delta_t^+$ . If  $\alpha \not\perp \beta_r$  where  $r < t$ , and  $\alpha + \beta_r$  is a root, then  $\alpha + \beta_r \in \Delta_s^+$  where  $s < r$ , and  $\langle \alpha + \beta_r, \beta_s \rangle > 0$ . That is impossible because  $\alpha \perp \beta_s \perp \beta_r$ . If  $\alpha \not\perp \beta_r$  now  $\beta_r - \alpha$  is a root. It is positive by the maximality property of  $\beta_r$ , so  $\alpha \in \Delta_r^+$ , contradicting  $\alpha \in \Delta_t^+$  with  $r < t$ . Thus  $\alpha \perp \beta_r$  for all  $r < t$ . As argued before,  $\alpha + \beta_t$  is not a root. Since  $\beta_t - \alpha \in \Delta^+$  now  $\langle \alpha, \beta_t \rangle > 0$ . That completes the induction.

Finally, let  $\alpha \in \Delta_r^+ \cup \{\beta_r\}$ ,  $\gamma \in \Delta_s^+ \cup \{\beta_s\}$ , and  $t = \min\{r, s\}$ . Suppose that  $\alpha + \gamma$  is a root. If  $i < t$  then  $\langle \alpha + \gamma, \beta_i \rangle = \langle \alpha, \beta_i \rangle + \langle \gamma, \beta_i \rangle = 0$ , and  $\langle \alpha + \gamma, \beta_t \rangle > 0$  because at least one of  $\langle \alpha, \beta_t \rangle$  and  $\langle \gamma, \beta_t \rangle$  is positive.  $\square$

Lemma 6.3 shows that the Lie algebra  $\mathfrak{n}$  of  $N$  is the vector space direct sum of its subspaces

$$(6.5) \quad \mathfrak{m}_r = \mathfrak{g}_{\beta_r} + \sum_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leq r \leq m$$

and Lemma 6.4 shows that  $\mathfrak{n}$  has an increasing foliation by ideals

$$(6.6) \quad \mathfrak{n}_r = \mathfrak{m}_1 + \mathfrak{m}_2 + \cdots + \mathfrak{m}_r \text{ for } 1 \leq r \leq m.$$

Now we will see that the corresponding group level decomposition  $N = M_1 M_2 \cdots M_m$  and the semidirect product decompositions  $N_r = N_{r-1} \rtimes M_r$  satisfy all the requirements of (5.1).

The structure of  $\Delta_r^+$ , and later of  $\mathfrak{m}_r$ , is exhibited by a particular Weyl group element of  $\Delta(\mathfrak{g}, \mathfrak{a})$ . Denote

$$(6.7) \quad s_{\beta_r} \text{ is the Weyl group reflection in } \beta_r \text{ and } \sigma_r : \Delta(\mathfrak{g}, \mathfrak{a}) \rightarrow \Delta(\mathfrak{g}, \mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha).$$

Note that  $\sigma_r(\beta_s) = -\beta_s$  for  $s \neq r$ ,  $+\beta_s$  if  $s = r$ . If  $\alpha \in \Delta_r^+$  we still have  $\sigma_r(\alpha) \perp \beta_i$  for  $i < r$  and  $\langle \sigma_r(\alpha), \beta_r \rangle > 0$ . If  $\sigma_r(\alpha)$  is negative then  $\beta_r - \sigma_r(\alpha) > \beta_r$  contradicting the maximality property of  $\beta_r$ . Thus, using Lemma 6.4,  $\sigma_r(\Delta_r^+) = \Delta_r^+$ .

**Lemma 6.8.** *If  $\alpha \in \Delta_r^+$  then  $\alpha + \sigma_r(\alpha) = \beta_r$ . (Of course it is possible that  $\alpha = \sigma_r(\alpha) = \frac{1}{2}\beta_r$  when  $\frac{1}{2}\beta_r$  is a root.). If  $\alpha, \alpha' \in \Delta_r^+$  and  $\alpha + \alpha' \in \Delta(\mathfrak{g}, \mathfrak{a})$  then  $\alpha + \alpha' = \beta_r$ .*

*Proof.* If  $\alpha \in \Delta_r^+$  with  $\sigma_r(\alpha) = \alpha$  then  $s_{\beta_r}(\alpha) = -\alpha$  so  $\alpha$  is proportional to  $\beta_r$ . As  $\beta_r$  is nonmultipliable and  $\langle \alpha, \beta_r \rangle > 0$  that forces  $\alpha = \frac{1}{2}\beta_r$ . In particular  $\alpha + \sigma_r(\alpha) = \beta_r$ .

Now suppose  $\alpha \in \Delta_r^+$  with  $\sigma_r(\alpha) \neq \alpha$ . Then  $\alpha + \sigma_r(\alpha) = \alpha - s_{\beta_r}(\alpha) = \alpha - (\alpha - \frac{2\langle \alpha, \beta_r \rangle}{\langle \beta_r, \beta_r \rangle} \beta_r) = \frac{2\langle \alpha, \beta_r \rangle}{\langle \beta_r, \beta_r \rangle} \beta_r$ . As  $\langle \alpha, \beta_r \rangle > 0$  and  $\beta_r$  is nonmultipliable this forces  $\alpha + \sigma_r(\alpha) = \beta_r$ .

Suppose that there exist  $\alpha, \alpha' \in \Delta_r^+$  such that  $\alpha + \alpha' = \alpha'' \in \Delta(\mathfrak{g}, \mathfrak{a})$  but  $\alpha'' \neq \beta_r$ . Fix such a pair  $\{\alpha, \alpha'\}$  with  $\alpha$  maximal for that property. Then  $\alpha''$  lacks that property. So  $\beta_r = \alpha'' + \sigma_r(\alpha'') = (\alpha + \alpha') + \sigma_r(\alpha + \alpha') = (\alpha + \sigma_r(\alpha)) + (\alpha' + \sigma_r(\alpha')) = 2\beta_r$ . Thus the specified  $\alpha$  cannot exist.  $\square$

Now we are in a position to start the proof of the main technical result of this section – that the  $M_r$  have square integrable representations. For that it suffices to consider the case where  $\mathfrak{g}$  is simple as a real Lie algebra and run through some possibilities:

**Lemma 6.9.** *Let  $\mathfrak{n}$  be a nilpotent Lie algebra,  $\mathfrak{z}$  its center, and  $\mathfrak{v}$  a vector space complement to  $\mathfrak{z}$  in  $\mathfrak{n}$ . Suppose that we have vector space direct sum decompositions  $\mathfrak{v} = \mathfrak{u} + \mathfrak{u}'$ ,  $\mathfrak{u} = \sum \mathfrak{u}_a$  and  $\mathfrak{u}' = \sum \mathfrak{u}'_a$ , and  $\mathfrak{z} = \sum \mathfrak{z}_b$  with  $\dim \mathfrak{z}_b = 1$ . Suppose further that (i) each  $[\mathfrak{u}_a, \mathfrak{u}_a] = 0 = [\mathfrak{u}'_a, \mathfrak{u}'_a]$ , (ii) if  $a_1 \neq a_2$  then  $[\mathfrak{u}_{a_1}, \mathfrak{u}'_{a_2}] = 0$  and (iii) for each index  $a$  there is a unique  $b_a$  such that  $\mathfrak{u}_a \otimes \mathfrak{u}'_a \rightarrow \mathfrak{z}_{b_a}$ , by  $u \otimes u' \mapsto [u, u']$ , is a nondegenerate pairing. Then  $\mathfrak{n}$  is a direct sum of Heisenberg algebras  $\mathfrak{z}_{b_a} + \mathfrak{u}_a + \mathfrak{u}'_a$  and the commutative algebra that is the sum of the remaining  $\mathfrak{z}_b$ .*

**Lemma 6.10.** *If  $\mathfrak{g}$  is the split real form of  $\mathfrak{g}_{\mathbb{C}}$  then each  $M_r$  has square integrable representations.*

**Lemma 6.11.** *If  $\mathfrak{g}$  is simple but not absolutely simple then each  $M_r$  has square integrable representations.*

**Lemma 6.12.** *If  $G$  is the quaternion special linear group  $SL(n; \mathbb{H})$  then  $M_1$  has square integrable representations.*

**Lemma 6.13.** *If  $G$  is the group  $E_{6, F_4}$  of collineations of the Cayley projective plane then  $M_1$  has square integrable representations.*

**Lemma 6.14.** *The group  $M_1$  has square integrable representations.*

**Lemma 6.15.** *If  $\mathfrak{g}$  is absolutely simple then each  $M_r$  has square integrable representations.*

*Proof.* (Lemma 6.9.) The assertion is obvious.  $\square$



*Proof.* (Lemma 6.10.) This is the case where  $\mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}$  and  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$  consists of the  $\mathbb{C}$ -linear extensions of the roots in  $\Delta(\mathfrak{g}, \mathfrak{a})$ . All roots are indivisible, so Lemma 6.8 divides  $\Delta_r^+$  into two disjoint subsets, thus divides  $\sum_{\alpha \in \Delta_r^+} \mathfrak{g}_{\alpha}$  as a direct sum  $\mathfrak{u} \oplus \mathfrak{u}'$  of two subspaces, such that those subspaces satisfy the conditions of Lemma 6.9. As  $\mathfrak{m}_r$  has 1-dimensional center  $\mathfrak{g}_{\beta_r}$  it follows that  $\mathfrak{m}_r$  is a Heisenberg algebra. Now  $M_r$  is a Heisenberg group and thus has square integrable representations.  $\square$

*Proof.* (Lemma 6.11.) This is the case where  $\mathfrak{g}$  is the underlying real structure of a complex simple Lie algebra  $\mathfrak{s}$ . The Cartan subalgebra  $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$  of  $\mathfrak{s}$  is given by  $\mathfrak{t} = \sqrt{-1}\mathfrak{a}$ , and  $\mathfrak{a}$  is the (real) subspace on which the roots take real values. As a real Lie algebra,  $\mathfrak{n} \cong \sum_{\alpha \in \Delta^+(\mathfrak{s}, \mathfrak{a})} \mathfrak{s}_{\alpha}$ .

Let  $\mathfrak{g}'$  denote the split real form of  $\mathfrak{s}$ . In the Iwasawa decomposition  $G' = K'A'N'$  now  $\mathfrak{a}' = \mathfrak{a}$ ,  $\mathfrak{n}'$  is a real form of  $\mathfrak{n}$ , and for each  $r$  the algebra  $\mathfrak{m}'_r := \mathfrak{m}_r \cap \mathfrak{n}'$  is a real form of  $\mathfrak{m}_r$ . From the latter, [11, Theorem 2.1] says that the corresponding group  $M'_r$  has square integrable representations if and only if its complexification  $M_r$  has square integrable representations. However, Lemma 6.10 says that  $M'_r$  has square integrable representations. Our assertion follows.  $\square$

*Proof.* (Lemma 6.12.) This is the case where  $\mathfrak{g} = \mathfrak{sl}(n; \mathbb{H})$ . In the usual root ordering,  $\tilde{\beta}_1 = \psi_1 + \dots + \psi_{2n-1}$  and  $\psi_i|_{\mathfrak{a}} = 0$  just when  $i$  is odd. Thus the ordinary roots that restrict to  $\beta_1$  are  $\psi_1 + \dots + \psi_{2n-1}$ ,  $\psi_2 + \dots + \psi_{2n-1}$ ,  $\psi_1 + \dots + \psi_{2n-2}$  and  $\psi_2 + \dots + \psi_{2n-2}$ ; their root spaces span the center  $\mathfrak{z}_1$  of  $\mathfrak{m}_1$ . Further  $\Delta_1^+$  consists of the restrictions of pairs of roots that

- sum to  $\psi_1 + \dots + \psi_{2n-1}$ :  $\{\psi_1 + \dots + \psi_j, \psi_{j+1} + \dots + \psi_{2n-1}\}$ ,  $1 \leq j < 2n-1$ ,
- sum to  $\psi_2 + \dots + \psi_{2n-1}$ :  $\{\psi_2 + \dots + \psi_j, \psi_{j+1} + \dots + \psi_{2n-1}\}$ ,  $2 \leq j < 2n-1$ ,
- sum to  $\psi_1 + \dots + \psi_{2n-2}$ :  $\{\psi_1 + \dots + \psi_j, \psi_{j+1} + \dots + \psi_{2n-2}\}$ ,  $1 \leq j < 2n-2$ ,
- sum to  $\psi_2 + \dots + \psi_{2n-2}$ :  $\{\psi_2 + \dots + \psi_j, \psi_{j+1} + \dots + \psi_{2n-2}\}$ ,  $2 \leq j < 2n-2$ .

Their root spaces span a complement  $\mathfrak{v}_1$  to  $\mathfrak{z}_1$  in  $\mathfrak{m}_1$ . Eliminating duplicates, the set of ordinary roots that restrict to elements of  $\Delta_1^+$  is  $\{\psi_1 + \dots + \psi_j; \psi_{j+1} + \dots + \psi_{2n-1}; \psi_2 + \dots + \psi_j; \psi_{j+1} + \dots + \psi_{2n-2}\}$ . Now let  $\lambda \in \mathfrak{z}_1^*$  be zero on the root spaces for  $\psi_2 + \dots + \psi_{2n-1}$  and  $\psi_1 + \dots + \psi_{2n-2}$ , nonzero on the root spaces for  $\psi_1 + \dots + \psi_{2n-1}$  and  $\psi_2 + \dots + \psi_{2n-2}$ . Then the corresponding antisymmetric bilinear form  $b_{\lambda}$  on  $\mathfrak{v}_1$  is nonsingular. Thus  $M_1$  has square integrable (modulo its center) representations.  $\square$

*Proof.* (Lemma 6.13.) This is the case where  $\mathfrak{g} = \mathfrak{e}_{6,F_4}$ . Then  $\text{rank}_{\mathbb{R}} \mathfrak{g} = 2$ . In the Bourbaki order for the

simple roots  $\begin{array}{cccccc} \psi_1 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ & & | & & \\ & & \psi_2 & & \end{array}$ ,  $\tilde{\beta}_1 = \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$ , and the roots that restrict to 0 on  $\mathfrak{a}$  are  $\psi_2, \psi_3, \psi_4$  and  $\psi_5$ . So  $\Delta_1^+$  consists of the restrictions of pairs of roots that

- sum to  $\psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$ :
  - $\{\psi_2, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_4, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_3 + \psi_4, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6\}$ ,
  - $\{\psi_1 + \psi_2 + \psi_3 + \psi_4, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5\}$ ,
  - $\{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\}$ ,
  - $\{\psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5\}$

- [illegible]

- sum to  $\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6$ :

$$\begin{aligned} & \{\{\psi_1, \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_6, \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5\}, \\ & \{\psi_1 + \psi_3, \psi_2 + \psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_2, \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + \psi_4\}\} \end{aligned}$$

- sum to  $\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$ :

$$\begin{aligned} & \{\{\psi_1, \psi_3 + \psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_3, \psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_3 + \psi_4, \psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5, \psi_6\}\} \end{aligned}$$

Eliminating duplicates, the set of ordinary roots that restrict to elements of  $\Delta_1^+$  consists of

- The 20 positive roots listed above in the first group, summing to  $\psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$ . These are the roots  $\sum a_i \psi_i$  for which  $a_2 = 1$  and  $(a_1, a_6)$  is either  $(1, 0)$  or  $(0, 1)$ . We denote the sum of their root spaces by  $\mathfrak{v}'_1$ .
- The 8 positive roots listed above in the last group, summing to  $\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$ . These are the roots  $\sum a_i \psi_i$  for which  $a_2 = 0$  and  $(a_1, a_6)$  is either  $(1, 0)$  or  $(0, 1)$ . We denote the sum of their root spaces by  $\mathfrak{v}''_1$ .

Now the space  $\mathfrak{v}_1 := \mathfrak{v}'_1 + \mathfrak{v}''_1$  is a vector space complement to  $\mathfrak{z}_1$  in  $\mathfrak{m}_1$ . Let  $\lambda \in \mathfrak{z}_1^*$  be zero on the root spaces for  $\psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$ ,  $\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6$ ,  $\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6$ ,  $\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6$ ,  $\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6$  and  $\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6$ , and nonzero on the root spaces for  $\psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$  and  $\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$ . Then the corresponding antisymmetric bilinear form  $b_\lambda$  on  $\mathfrak{v}_1$  is nonsingular, so  $M_1$  has square integrable (modulo its center) representations.  $\square$

*Proof.* (Lemma 6.14.) It suffices to consider the case where  $\mathfrak{g}_\mathbb{C}$  is a simple complex Lie algebra, but  $\mathfrak{g}$  need not be its split real form. We do, however, assume that it is not the compact real form, for in that case  $N = \{1\}$ .

Suppose first that  $\dim \mathfrak{g}_{\beta_1} = 1$ . In other words the highest root in  $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ , call it  $\tilde{\beta}_1$ , is the only root of  $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$  that restricts to  $\beta_1$ . Applying Lemma 6.8 as in the proof of Lemma 6.10 it follows that  $M_1$  has square integrable representations.

Now suppose that  $\dim \mathfrak{g}_{\beta_1} > 1$ . Note that the roots in  $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$  that restrict to  $\beta_1$  are just the roots of the form  $\tilde{\beta}_1 - \sum t_i \gamma_i$  where every one of the  $\gamma_i|_{\mathfrak{a}} = 0$ . In particular the root(s) of the extended Dynkin diagram of  $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ , to which  $-\tilde{\beta}_1$  attaches, have restriction 0 on  $\mathfrak{a}$ . We have already dealt with the cases  $\mathfrak{g} = \mathfrak{sl}(n; \mathbb{H})$  and  $\mathfrak{g} = \mathfrak{e}_{6,F_4}$ , so there remain only a few easy cases:

Case  $\mathfrak{g} = \mathfrak{so}(1, n)$ . Then  $\text{rank}_{\mathbb{R}} \mathfrak{g} = 1$ ,  $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\tilde{\beta}_1\}$  and  $M_1 = N$  is abelian. In particular  $M_1$  has square integrable (modulo its center) representations.

Case  $\mathfrak{g} = \mathfrak{su}(1, n)$ . Then  $\text{rank}_{\mathbb{R}} \mathfrak{g} = 1$ ,  $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\tilde{\beta}_1, \frac{1}{2}\tilde{\beta}_1\}$ , and  $M_1 = N$  is a Heisenberg group  $\text{Im } \mathbb{C} + \mathbb{C}^{n-1}$ . In particular  $M_1$  has square integrable representations.

Case  $\mathfrak{g} = \mathfrak{sp}(p, q)$ ,  $p \leq q$ . Then  $\text{rank}_{\mathbb{R}} \mathfrak{g} = p$ ,  $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\tilde{\beta}_1, \frac{1}{2}\tilde{\beta}_1\}$ , and  $M_1 = N$  is a quaternionic Heisenberg group  $\text{Im } \mathbb{H} + \mathbb{H}^s$ . In particular  $M_1$  has square integrable (modulo its center) representations.

Case  $\mathfrak{g} = \mathfrak{f}_{4,B_4}$ . Then  $\text{rank}_{\mathbb{R}} \mathfrak{g} = 1$ ,  $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\tilde{\beta}_1, \frac{1}{2}\tilde{\beta}_1\}$ , and  $M_1 = N$  is an octonionic Heisenberg group  $\text{Im } \mathcal{O} + \mathcal{O}$ . In particular  $M_1$  has square integrable (modulo its center) representations.  $\square$

*Proof.* (Lemma 6.15.) This is the case where  $\mathfrak{g}_\mathbb{C}$  is a simple complex Lie algebra, but  $\mathfrak{g}$  need not be its split real form. We do, however, assume that it is not the compact real form, for in that case  $N = \{1\}$ . Then  $\beta_1(\mathfrak{t}) = 0$ . Note that  $\beta_1$  is the restriction to  $\mathfrak{a}$  of the highest root  $\tilde{\beta}_1$  in  $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$  and that  $\tilde{\beta}_1$  is

a long root. Thus  $\tilde{\beta}_1$  is the only root in  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  that restricts of  $\beta_1$ . Applying Lemma 6.8 as in the proof of Lemma 6.10 it follows that  $M_1$  has square integrable representations. This starts the induction.

Suppose we know that  $M_1, \dots, M_r$  have square integrable representations and that  $r < m$ . Let  $\mathfrak{g}_r$  be the semisimple subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{h}$  and all (restricted) root spaces for simple roots that are orthogonal to  $\beta_1, \beta_2, \dots, \beta_r$ . Then  $\beta_{r+1}$  is a maximum among the positive restricted roots of  $\mathfrak{g}_r$  and  $\mathfrak{m}_{r+1}$  is the subalgebra of  $\mathfrak{g}_r$  that is the counterpart of  $\mathfrak{m}_1$  for  $\mathfrak{g}$ . Thus by the argument just above for  $M_1$ , and by Lemmas 6.10 and 6.11 as needed for the simple summands of  $\mathfrak{g}_r$ , we conclude that  $M_{r+1}$  has square integrable representations.  $\square$

We now apply Lemmas 6.11 and 6.15 to the list (5.1) of conditions for setting up the character formula and Plancherel formula as in Theorem 5.3. Those Lemmas supply the key condition, that each  $M_r$  has unitary representations with coefficients in  $L^2(M_r/Z_r)$ . Lemma 6.4 ensures that each  $N_r := M_1 M_2 \dots M_r$  is a normal subgroup of  $N$  with  $N_r = N_{r-1} \rtimes M_r$  semidirect product, and then Lemma 6.2 says that  $N = M_1 M_2 \dots M_{m-1} M_m$  as needed. The decompositions  $\mathfrak{m}_r = \mathfrak{z}_r + \mathfrak{v}_r$  and  $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$  now are immediate from the construction of the  $\mathfrak{m}_r$ . It remains only to verify that  $[\mathfrak{m}_r, \mathfrak{z}_s] = 0$  and  $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{v}$  for  $r > s$ .

Let  $\alpha \in \Delta_r^+$  and  $s < r$ . Then  $\alpha \notin (\Delta_1^+ \cup \dots \cup \Delta_s^+)$  and  $\alpha \perp \beta_i$  for  $i \leq s$ . Now  $\beta_s + \alpha \in \Delta^+$  would imply  $\beta_s - \alpha \in \Delta^+$ , contradicting  $\alpha \notin \Delta_s^+$ . It follows that  $[\mathfrak{m}_r, \mathfrak{z}_s] = 0$ .

Let  $\alpha \in \Delta_r^+$  and  $\alpha' \in \Delta_s^+$ ,  $s < r$ , with  $\alpha + \alpha' = \beta_t$ . Lemma 6.4 says  $s = t$  so  $\beta_s - \alpha' = \alpha$ . But then  $\beta_s - \alpha' \in \Delta_s^+$  contradicting  $\alpha \notin (\Delta_1^+ \cup \dots \cup \Delta_s^+)$ . We conclude that  $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{v}$  for  $r > s$ .

Summarizing, we have just shown that Theorem 5.3 applies to nilradicals of minimal parabolic subgroups. In other words,

**Theorem 6.16.** *Let  $G$  be a real reductive Lie group,  $G = KAN$  an Iwasawa decomposition,  $\mathfrak{m}_r$  and  $\mathfrak{n}_r$  the subalgebras of  $\mathfrak{n}$  defined in (6.5) and (6.6), and  $M_r$  and  $N_r$  the corresponding analytic subgroups of  $N$ . Then the  $M_r$  and  $N_r$  satisfy (5.1). In particular, Plancherel measure for  $N$  is concentrated on  $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$ . If  $\lambda \in \mathfrak{t}^*$ , and if  $u$  and  $v$  belong to the representation space  $\mathcal{H}_{\pi_\lambda}$  of  $\pi_\lambda$ , then the coefficient  $f_{u,v}(x) = \langle u, \pi_\lambda(x)v \rangle$  satisfies*

$$(6.17) \quad \|f_{u,v}\|_{L^2(N/S)}^2 = \frac{\|u\|^2 \|v\|^2}{|P(\lambda)|}.$$

The distribution character  $\Theta_{\pi_\lambda}$  of  $\pi_\lambda$  satisfies

$$(6.18) \quad \Theta_{\pi_\lambda}(f) = c^{-1} |P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f}_1(\xi) d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N)$$

where  $\mathcal{C}(N)$  is the Schwartz space,  $f_1$  is the lift  $f_1(\xi) = f(\exp(\xi))$ ,  $\widehat{f}_1$  is its classical Fourier transform,  $\mathcal{O}(\lambda)$  is the coadjoint orbit  $\text{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$ , and  $d\nu_\lambda$  is the translate of normalized Lebesgue measure from  $\mathfrak{v}^*$  to  $\text{Ad}^*(N)\lambda$ . The Plancherel formula on  $N$  is

$$(6.19) \quad f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f) |P(\lambda)| d\lambda \text{ for } f \in \mathcal{C}(N).$$

## 7 Arithmetic Quotients

In this section we consider the case where our connected simply connected nilpotent Lie group  $N$  has a discrete co-compact subgroup  $\Gamma$  that fits into a decomposition of the form (5.1). We show that the compact nilmanifold  $N/\Gamma$  has a corresponding foliation and derive analytic results analogous to those of Theorem 5.3. These results include multiplicity formulae for the regular representation of  $N$  on  $L^2(N/\Gamma)$ . They apply in particular to the nilradicals of minimal parabolic subgroups, as studied in Section 6.

Here are some basic facts about discrete uniform (i.e. co-compact) subgroups of connected simply connected nilpotent Lie groups, mostly due to Mal'cev. See [10, Chapter 2] for an exposition.

**Proposition 7.1.** *Let  $N$  be a connected simply connected nilpotent Lie group. Then the following are equivalent.*

- $N$  has a discrete subgroup  $\Gamma$  with  $N/\Gamma$  compact.
- $N \cong N_{\mathbb{R}}$  where  $N_{\mathbb{R}}$  is the group of real points in a unipotent linear algebraic group defined over the rational number field  $\mathbb{Q}$
- $\mathfrak{n}$  has a basis  $\{\xi_j\}$  for which the coefficients  $c_{i,j}^k$  in  $[\xi_i, \xi_j] = \sum c_{i,j}^k \xi_k$  are rational numbers.

Under those conditions let  $\mathfrak{n}_{\mathbb{Q}}$  denote the rational span of  $\{\xi_j\}$  and let  $\mathfrak{n}_{\mathbb{Z}}$  be the integral span. Then  $\exp(\mathfrak{n}_{\mathbb{Z}})$  generates a discrete subgroup  $N_{\mathbb{Z}}$  of  $N = N_{\mathbb{R}}$  and  $N_{\mathbb{R}}/N_{\mathbb{Z}}$  is compact. Conversely, if  $\Gamma$  is a discrete co-compact subgroup of  $N$  then the  $\mathbb{Z}$ -span of  $\exp^{-1}(\Gamma)$  is a lattice in  $\mathfrak{n}$  for which any generating set  $\{\xi_j\}$  is a basis of  $\mathfrak{n}$  such that the coefficients  $c_{i,j}^k$  in  $[\xi_i, \xi_j] = \sum c_{i,j}^k \xi_k$  are rational numbers.

Note that the conditions of Proposition 7.1 hold for the nilpotent groups studied in Section 6, where in fact one can choose the basis  $\{\xi_j\}$  of  $\mathfrak{n}$  so that the  $c_{i,j}^k$  are integers.

Here are the basic facts on square integrable representations in this setting, from [8, Theorem 7]:

**Proposition 7.2.** *Let  $N$  be a connected simply connected nilpotent Lie group that has square integrable representations, and let  $\Gamma$  a discrete co-compact subgroup. Let  $Z$  be the center of  $N$  and normalize the volume form on  $\mathfrak{n}/\mathfrak{z}$  by normalizing Haar measure on  $N$  so that  $N/Z\Gamma$  has volume 1. Let  $P$  be the corresponding Pfaffian polynomial on  $\mathfrak{z}^*$ . Note that  $\Gamma \cap Z$  is a lattice in  $Z$  and  $\exp^{-1}(\Gamma \cap Z)$  is a lattice (denote it  $\Lambda$ ) in  $\mathfrak{z}$ . That defines the dual lattice  $\Lambda^*$  in  $\mathfrak{z}^*$ . Then a square integrable representation  $\pi_{\lambda}$  occurs in  $L^2(N/\Gamma)$  if and only if  $\lambda \in \Lambda^*$ , and in that case  $\pi_{\lambda}$  occurs with multiplicity  $|P(\lambda)|$ .*

**Definition 7.3.** Let  $N = N_{\mathbb{R}}$  be defined over  $\mathbb{Q}$  as in Proposition 7.1, so we have a fixed rational form  $N_{\mathbb{Q}}$ . We say that a connected Lie subgroup  $M \subset N$  is *rational* if  $M \cap N_{\mathbb{Q}}$  is a rational form of  $M$ , in other words if  $\mathfrak{m} \cap \mathfrak{n}_{\mathbb{Q}}$  contains a basis of  $\mathfrak{m}$ . We say that a decomposition (5.1) is *rational* if the subgroups  $M_r$  and  $N_r$  are rational.  $\diamond$

The following is immediate from this definition.

**Lemma 7.4.** *Let  $N$  be defined over  $\mathbb{Q}$  as in Proposition 7.1 with rational structure defined by a discrete co-compact subgroup  $\Gamma$ . If the decomposition (5.1) is rational then each  $\Gamma \cap Z_r$  in  $Z_r$ , each  $\Gamma \cap M_r$  in  $M_r$ , each  $\Gamma \cap S_r$  in  $S_r$ , and each  $\Gamma \cap N_r$  in  $N_r$ , is a discrete co-compact subgroup defining the same rational structure as the one defined by its intersection with  $N_{\mathbb{Q}}$ .*

For the rest of this section we will assume that  $N$  and  $\Gamma$  satisfy the rationality conditions of Lemma 7.4, in particular that (5.1) is rational. Then for each  $r$ ,  $Z_r \cap \Gamma$  is a lattice in the center  $Z_r$  of  $M_r$ , and  $\Lambda_r := \log(Z_r \cap \Gamma)$  is a lattice in its Lie algebra  $\mathfrak{z}_r$ . That defines the dual lattice  $\Lambda_r^*$  in  $\mathfrak{z}_r^*$ . We normalize the Pfaffian polynomials on the  $\mathfrak{z}_r^*$ , and thus the polynomial  $P$  on  $\mathfrak{s}^*$ , by requiring that the  $N_r/(S_r \cdot (N_r \cap \Gamma))$  have volume 1.

**Theorem 7.5.** *Let  $\lambda \in \mathfrak{t}^*$ , in other words  $\lambda = \sum \lambda_r$  where  $\lambda_r \in \mathfrak{z}_r^*$  with  $\text{Pf}(b_{\lambda_r}) \neq 0$ . Then a stepwise square integrable representation  $\pi_{\lambda}$  of  $N$  occurs in  $L^2(N/\Gamma)$  if and only if each  $\lambda_r \in \Lambda_r^*$ , and in that case the multiplicity of  $\pi_{\lambda}$  on  $L^2(N/\Gamma)$  is  $|P(\lambda)|$ .*

*Proof.* Recall  $N_r = M_1 M_2 \dots M_r = N_{r-1} \rtimes M_r$  semidirect product, where  $N = M_1 M_2 \dots M_m$  and the center  $Z_r$  of  $M_r$  is central in  $N_r$ . Fix  $r \leq m$ . By induction on dimension we assume that Theorem 7.5 holds for  $N_{r-1}$  and  $\Gamma \cap N_{r-1}$ . We may also assume that  $\dim Z_r = 1$ , following the argument of the first paragraph of the proof of [8, Theorem 7].

Now we proceed as in [8], adapted to our situation. Choose nonzero rational  $x \in \mathfrak{m}_r \setminus \mathfrak{z}_r$  and  $z \in \mathfrak{z}_r$  in such a way that (i)  $\exp(z)$  generates the infinite cyclic group  $\Gamma \cap Z_r$ , (ii)  $[x, \mathfrak{m}_r] \subset \mathfrak{z}_r$ , and (iii)  $\exp(x)$  and  $\exp(z)$  generate  $\Gamma \cap P_r$  where  $P_r = \exp(\mathfrak{p}_r)$  where  $\mathfrak{p}_r$  is the span of  $x$  and  $z$ . The centralizer  $Z_{M_r}(x)$  of  $x$  in  $M_r$  is a rational normal subgroup of codimension 1 in  $M_r$ , so  $Q_r := N_{r-1} \rtimes Z_{M_r}(x)$  is a rational normal subgroup of codimension 1 in  $N_r$ . The group  $\Gamma Q_r/Q_r$  is infinite cyclic. Parameterize  $\mathfrak{z}_r^*$  by  $a = a(\nu_r) = \nu_r(z)$ . The Pfaffian polynomial on  $\mathfrak{z}_r^*$ , normalized by the condition that  $M_r/\Gamma Z_r$  has volume 1, satisfies  $P_r(a) = \text{Pf}(b_{\nu_r}) = c_r a^{d_r}$  where  $\nu_r(z) = a$ ,  $\dim M_r = 2d_r + 1$ , and  $c_r$  is a nonzero constant.

Choose  $\gamma \in \Gamma$  whose image in  $\Gamma Q_r/Q_r$  is a generator and let  $y = \log(\gamma)$ . Then  $[x, y]$  is a rational multiple of  $x$ , say  $[x, y] = uz$ . Since  $\exp(x)$ ,  $\exp(y)$  and  $\exp(z)$  span a rational 3-dimensional Heisenberg algebra which we denote  $\mathfrak{h}_r$ ;  $H_r$  denotes the corresponding group. It follows [1] that  $u$  is an integer.

Let  $\pi_\nu \in \widehat{N_r}$  occur in  $L^2(N_r/(\Gamma \cap N_r))$  where  $\nu = \nu_1 + \dots + \nu_r$  with  $\nu_i \in \mathfrak{z}_i$  and  $\text{Pf}(b_{\nu_i}) \neq 0$ . By induction,  $\nu_i \in \Lambda^*$  for  $i < r$ , and the argument immediately above shows that  $\nu_r \in \Lambda^*$ . In other words, by induction on dimension and on  $r$ , if  $\pi_\lambda$  occurs on  $L^2(N/\Gamma)$  then for each index  $i$  we have  $\lambda_i \in \Lambda_i^*$ .

By construction,  $Q_r$  satisfies (5.1) with  $Z_{M_r}(x)$  in place of  $M_r$ . If  $\nu_r \in \mathfrak{z}_r^*$  defines a square integrable (mod  $Z_r$ ) representation of  $M_r$ , then it also defines a square integrable (mod  $P_r$ ) representation of  $Z_{M_r}(x)$ . Let  $\xi_\nu \in \widehat{Q_r}$  correspond to  $\nu = \nu_1 + \dots + \nu_r$  with each  $\nu_i \in \Lambda^* \cap \mathfrak{z}_i^*$  and  $\text{Pf}(b_{\nu_i}) \neq 0$ . By induction on dimension we may assume that  $\xi_\nu$  has multiplicity  $|\text{Pf}(b_{\nu_1}) \dots \text{Pf}(b_{\nu_{r-1}}) \text{Pf}'(b_{\nu_r})|$  on  $L^2(Q_r/(\Gamma \cap Q_r))$ , where  $\text{Pf}'(b_{\nu_r})$  is the Pfaffian computed on the Lie algebra of  $Z_{M_r}(x)$  (modulo its center  $\mathfrak{p}_r$ ).

The square integrable representations of  $Z_{M_r}(x)$  are parameterized by the linear functionals  $\mu_r$  on  $\mathfrak{p}_r = \mathfrak{z}_r + x\mathbb{R}$  with  $\text{Pf}' \neq 0$ . We parameterize  $\mu_r$  by  $a = \mu_r(z)$  and  $b = \mu_r(x)$  so  $\text{Pf}'(b_{\mu_r})$  is a polynomial in  $a$  and  $b$ . By construction it is independent of  $x$ , so  $\text{Pf}'(b_{\mu_r}) = c'_r a^{d_r-1}$  where  $c'_r$  is a constant and  $\dim \mathfrak{m}_r/\mathfrak{z}_r = 2d_r$ . Define  $\nu_r$  by  $a = \nu_r(z)$ , i.e. by  $\nu_r = \mu_r|_{\mathfrak{z}_r}$ . Since  $[x, y] = uz$  now  $\text{Pf}(a) = \text{Pf}'(a, b)au = uc'_r a^{d_r}$ , in particular  $c_r = uc'_r$ .

By induction on dimension,  $\xi_\mu$  occurs in  $L^2(Q_r/(\Gamma \cap Q_r))$  if and only if each  $\mu_i \in \Lambda^* \cap \mathfrak{z}_i^*$  with  $\text{Pf}(b_{\mu_i}) \neq 0$  for  $i < r$ , both  $a$  and  $b$  are integers, and  $a \neq 0$ . To simplify the notation, fix the  $\mu_i$  for  $i < r$  and write  $\xi(a, b)$  for the  $\xi_\mu$  where  $\mu_r$  has parameter  $(a, b)$ . Then  $\xi(a, b)$  has multiplicity  $\text{mult}'(a, b) = |\text{Pf}(b_{\mu_1}) \dots \text{Pf}(b_{\mu_{r-1}}) \text{Pf}'(b_{\mu_r})| = |\text{Pf}(b_{\mu_1}) \dots \text{Pf}(b_{\mu_{r-1}}) c'_r a^{d_r-1}|$ . Thus  $c'_r$  is an integer, so  $c_r = uc'_r$  is an integer as well.

Note that  $\pi_\nu = \text{Ind}_{Q_r}^{L_r}(\xi_\mu)$  whenever  $\mu|_{\mathfrak{z}_r} = \nu$  and that  $\pi_\nu|_{Q_r}$  is the direct integral of all such  $\xi_\mu$ . Denote  $A'(\nu) = \{\mu \mid \mu|_{\mathfrak{z}_r} = \nu \text{ and } \xi_\mu \text{ occurs in } L^2(Q_r/(\Gamma \cap Q_r))\}$ . It consists of all  $\xi(a, b)$  with fixed  $a = \nu_r(z) \neq 0$  and integral  $b$  if  $a$  is an integer, the empty set if  $a$  is not integral. Fix a set  $A(\nu)$  of representatives of the orbits of  $\Gamma \cap N_r$  on  $A'(\nu)$ . As in the proof of [8, Theorem 7], the algorithm of [7, page 153] says that the multiplicity of  $\pi_\nu$  in  $L^2(L_r/(\Gamma \cap L_r))$  is  $\text{mult}(\nu) = \sum_{\mu \in A(\nu)} \text{mult}'(\mu)$ .

An immediate consequence:  $\text{mult}(\nu) > 0$  if and only if each  $\nu_i \in \Lambda^*$ . That proves the first assertion of the Theorem.

We look at action of  $\Gamma \cap N_r$  on  $A'(\nu)$ . First,  $\Gamma \cap Q_r$  acts trivially, so the action is given by the cyclic group  $(\Gamma \cap N_r)/(\Gamma \cap Q_r)$ , which has generator  $\bar{\gamma} = \exp(y)(\Gamma \cap Q_r)$ . As  $[x, y] = uz$  the action is  $\bar{\gamma} : \xi(a, b) \mapsto \xi(a, b + au)$ . So we can assume that  $A(\nu)$  consists of the  $au$  elements  $\xi(a, b + i)$  where  $i$  is integral with  $0 \leq i < au$ . Each  $\text{mult}'(a, b + i) = |\text{Pf}(b_{\mu_1}) \dots \text{Pf}(b_{\mu_{r-1}}) c'_r a^{d_r-1}|$ , so now  $\text{mult}(\nu) = |\text{Pf}(b_{\nu_1}) \dots \text{Pf}(b_{\nu_{r-1}})| \cdot |auc'_r a^{d_r-1}| = |\text{Pf}(b_{\nu_1}) \dots \text{Pf}(b_{\nu_{r-1}})| \cdot |\text{Pf}(b_{\nu_r})|$ . This completes the proof of the induction step, and thus of the Theorem.  $\square$

## 8 Appendix: Some Concrete Calculations

This Appendix will be left in the arXiv version of the paper but removed before submitting the paper for publication.

Here are concrete examples demonstrating the root structure of the key technical tool, Lemma 6.8, behind Theorem 6.16. These are the cases where  $\Delta(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a})$ , i.e. where no restricted roots are divisible. The modification to allow divisible restricted roots is straightforward.

First suppose that  $\Delta$  is of type A  $\xrightarrow{\psi_1} \xrightarrow{\psi_2} \xrightarrow{\psi_{\ell-2}} \dots \xrightarrow{\psi_{\ell-1}}$ . Then  $r \leq [\ell/2]$  and  $\beta_r = \psi_r + \psi_{r+1} + \dots + \psi_{\ell-r}$ . If  $\alpha, \alpha' \in \Delta^+$  with  $\alpha + \alpha' = \beta_r$  then one of  $\alpha, \alpha'$  must have form  $\psi_r + \dots + \psi_s$  and then the other must be  $\psi_{s+1} + \dots + \psi_{\ell-r}$ . The assertions of Lemma 6.8 follow by inspection.

Now suppose that  $\Delta$  is of type B  $\xrightarrow{\psi_1} \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{n-1}} \xrightarrow{\psi_n}$ . Then  $\beta_1 = \psi_1 + 2(\psi_2 + \dots + \psi_n)$ ,  $\beta_2 = \psi_1$ ,  $\beta_3 = \psi_3 + 2(\psi_4 + \dots + \psi_n)$ ,  $\beta_4 = \psi_3$ , etc. If  $r$  is even,  $\beta_r = \psi_{r-1}$ , then  $\Delta_r^+$  is empty and the assertion is trivial. Now let  $r$  be odd,  $\beta_r = \psi_r + 2(\psi_{r+1} + \dots + \psi_n)$ . If  $\alpha, \alpha' \in \Delta^+$  with  $\alpha + \alpha' = \beta_r$  then one of  $\alpha$  must have form  $\sum a_i \psi_i$  and  $\alpha'$  must have form  $\sum a'_i \psi_i$  with  $a_{r+1} = 1 = a'_{r+1}$ ,  $a_i = 0 = a'_i$  for  $i < r$ , and one of  $a_r, a'_r$  is 1 while the other is 0. Now we may suppose

$$\alpha = \psi_r + \psi_{r+1} + \dots + \psi_u \text{ or } \alpha = \psi_r + \psi_{r+1} + \dots + \psi_u + 2(\psi_{u+1} + \dots + \psi_n)$$

and

$$\alpha' = \psi_{r+1} + \cdots + \psi_{u'} \text{ or } \alpha' = \psi_{r+1} + \cdots + \psi_{u'} + 2(\psi_{u'+1} + \cdots + \psi_n)$$

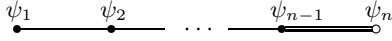
If neither of the 2's appears we must have

$$\alpha = \psi_r + \psi_{r+1} + \cdots + \psi_n \text{ and } \alpha' = \psi_{r+1} + \cdots + \psi_n.$$

The coefficient 2 cannot occur in both  $\alpha$  and  $\alpha'$  because  $\psi_n$  has coefficient 2 in  $\beta_r$ . That leaves the possibilities

$$\alpha = \psi_r + \psi_{r+1} + \cdots + \psi_u \text{ and } \alpha' = \psi_{r+1} + \cdots + \psi_{u'} + 2(\psi_{u'+1} + \cdots + \psi_n)$$

or the same with  $\alpha$  and  $\alpha'$  interchanged. In these cases  $u = u' < n$ . More generally the argument shows that if  $\alpha + \alpha'$  is a root then  $\alpha + \alpha' = \beta_r$ .

Next suppose that  $\Delta$  is of type  $C$   Then  $\beta_1 = 2(\psi_1 + \cdots + \psi_{n-1} + \psi_n)$ ,  $\beta_2 = 2(\psi_2 + \cdots + \psi_{n-1} + \psi_n)$ , etc. If  $\alpha, \alpha' \in \Delta_r^+$  with  $\alpha + \alpha' = \beta_r$  then one of  $\alpha$  must have form  $\sum a_i \psi_i$  and  $\alpha'$  must have form  $\sum a'_i \psi_i$  with  $a_r = 1 = a'_r$ ,  $a_i = 0 = a'_i$  for  $i < r$ , and one of  $a_n, a'_n$  is 1 while the other is 0. Now we may suppose

$$\alpha = \psi_r + \psi_{r+1} + \cdots + \psi_u \text{ or } \alpha = \psi_r + \psi_{r+1} + \cdots + \psi_u + 2(\psi_{u+1} + \cdots + \psi_{n-1}) + \psi_n$$

and

$$\alpha' = \psi_r + \psi_{r+1} + \cdots + \psi_{u'} \text{ or } \alpha' = \psi_r + \cdots + \psi_{u'} + 2(\psi_{u'+1} + \cdots + \psi_{n-1}) + \psi_n$$

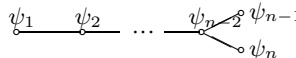
If the coefficient 2 appears in neither then

$$\alpha = \psi_r + \psi_{r+1} + \cdots + \psi_u \text{ and } \alpha' = \psi_r + \psi_{r+1} + \cdots + \psi_{u'}$$

with one of  $u, u'$  equal to  $n$  and the other equal to  $n-1$ . The coefficient 2 cannot appear in both because  $\psi_n$  has coefficient 1 in  $\beta_r$ . So now, interchanging  $\alpha$  and  $\alpha'$  if necessary,

$$\alpha = \psi_r + \psi_{r+1} + \cdots + \psi_u \text{ and } \alpha' = \psi_r + \cdots + \psi_{u'} + 2(\psi_{u'+1} + \cdots + \psi_{n-1}) + \psi_n.$$

For their sum to be a root we must have  $u = u' < r$ .

Now suppose that  $\Delta$  is of type  $D$   Then  $\beta_1 = \psi_1 + 2(\psi_2 + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$ ,  $\beta_2 = \psi_1$ ,  $\beta_3 = \psi_3 + 2(\psi_4 + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$ ,  $\beta_4 = \psi_3$ , etc, as for type  $B$ . If  $r$  is even then  $\beta_r = \psi_{r-1}$  and  $\Delta_r^+$  is empty. Now suppose that  $r$  is odd so  $\beta_r = \psi_r + 2(\psi_{r+1} + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$ . If  $\alpha, \alpha' \in \Delta_r^+$  with  $\alpha + \alpha' = \beta_r$  now  $\alpha = \sum a_i \psi_i$  and  $\alpha' = \sum a'_i \psi_i$  with  $a_i = 0 = a'_i$  for  $i < r$ ,  $a_{r+1} = 1 = a'_{r+1}$ , and for  $t \in \{r, n-1, n\}$  one of  $a_t, a'_t$  is 0 and the other is 1. We may and do suppose that  $a_n = 0$  and  $a'_n = 1$ . Then

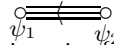
$$\alpha = a_r \psi_r + (\psi_{r+1} + \cdots + \psi_u) \text{ with } r+1 \leq u \leq n-1$$

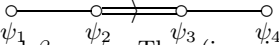
and

$$\alpha' = (1 - a_r) \psi_r + (\psi_{r+1} + \cdots + \psi_u) + 2(\psi_{u+1} + \cdots + \psi_{n-2}) + (1 - a_{n-1}) \psi_{n-1} + \psi_n$$

If  $u = n-1$  then  $1 - a_{n-1} = 0$  and the  $\alpha'$  just above is not a root, so  $r+1 \leq u \leq n-2$ . In particular  $\alpha + \alpha' \in \Delta^+$  implies  $\alpha + \alpha' = \beta_r$ .

We have finished with the classical structures and go on to the exceptional ones.

Suppose that  $\Delta$  is of type  $G_2$   . Then  $\beta_1 = 3\psi_1 + 2\psi_2$  and  $\beta_2 = \psi_1$ . So  $\Delta_1^+ = \{3\psi_1 + \psi_2, \psi_2\}$ ,  $\{2\psi_1 + \psi_2, \psi_1 + \psi_2\}$  in pairs  $\{\alpha, \alpha'\}$  with sum  $\beta_1$ , and  $\Delta_2^+$  is empty. The assertions of Lemma 6.8 follow by inspection.

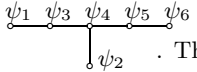
Suppose that  $\Delta$  is of type  $F_4$  . Then  $\beta_1 = 2\psi_1 + 3\psi_2 + 4\psi_4 + 2\psi_4$ ,  $\beta_2 = \psi_2 + 2\psi_3 + 2\psi_4$ ,  $\beta_3 = \psi_2 + 2\psi_3$  and  $\beta_4 = \psi_2$ . Thus (in pairs  $\{\alpha, \alpha'\}$  with sum  $\beta_1$ )

$$\begin{aligned} \Delta_1^+ = \{ & \{\psi_1, \psi_1 + 3\psi_2 + 4\psi_3 + 2\psi_4\}, \{\psi_1 + \psi_2, \psi_1 + 2\psi_2 + 4\psi_3 + 2\psi_4\}, \\ & \{\psi_1 + \psi_2 + \psi_3, \psi_1 + 2\psi_2 + 3\psi_3 + 2\psi_4\}, \{\psi_1 + \psi_2 + 2\psi_3, \psi_1 + 2\psi_2 + 2\psi_3 + 2\psi_4\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4, \psi_1 + 2\psi_2 + 3\psi_3 + \psi_4\}, \{\psi_1 + 2\psi_2 + 2\psi_3, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4\}, \\ & \{\psi_1 + \psi_2 + 2\psi_3 + \psi_4, \psi_1 + 2\psi_2 + 2\psi_3 + \psi_4\}\}. \end{aligned}$$

Similarly,

$$\Delta_2^+ = \{\{\psi_4, \psi_2 + 2\psi_3 + \psi_4\}, \{\psi_3 + \psi_4, \psi_2 + \psi_3 + \psi_4\}\}, \Delta_3^+ = \{\psi_3, \psi_2 + \psi_3\} \text{ and } \Delta_4^+ = \emptyset$$

In each case if  $\alpha, \alpha' \in \Delta_r^+$  and  $\alpha + \alpha'$  is a root then  $\alpha + \alpha' = \beta_r$ . The assertions of Lemma 6.8 follow by inspection.

Suppose that  $\Delta$  is of type  $E_6$  . Then the strongly orthogonal roots  $\beta_i$  are given by

$$\begin{aligned} \beta_1 &= \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6, \\ \beta_2 &= \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \\ \beta_3 &= \psi_3 + \psi_4 + \psi_5, \text{ and} \\ \beta_4 &= \psi_4. \end{aligned}$$

Now

$$\begin{aligned} \Delta_1^+ = \{ & \{\psi_2, \psi_1 + \psi_2 + 2\psi_2 + 3\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_4, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_3 + \psi_4, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5\}\} \end{aligned}$$

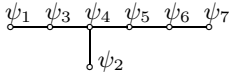
and

$$\Delta_2^+ = \{\{\psi_1, \psi_3 + \psi_4 + \psi_5 + \psi_6; \psi_6, \psi_1 + \psi_3 + \psi_4 + \psi_5\}; \\ \{\psi_1 + \psi_3, \psi_4 + \psi_5 + \psi_6, \psi_5 + \psi_6, \psi_1 + \psi_3 + \psi_4\}\}$$

while

$$\Delta_3^+ = \{\{\psi_3, \psi_4 + \psi_5\}, \{\psi_5, \psi_3 + \psi_4\}\} \text{ and } \Delta_4^+ = \emptyset.$$

In each case if  $\alpha, \alpha' \in \Delta_r^+$  and  $\alpha + \alpha'$  is a root then  $\alpha + \alpha' = \beta_r$ . The assertions of Lemma 6.8 follow by inspection..

Suppose that  $\Delta$  is of type  $E_7$  . Then the strongly orthogonal roots  $\beta_i$  are

$$\begin{aligned} \beta_1 &= 2\psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7, \\ \beta_2 &= \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7, \\ \beta_3 &= \psi_7, \\ \beta_4 &= \psi_2 + \psi_3 + 2\psi_4 + \psi_5, \\ \beta_5 &= \psi_2, \\ \beta_6 &= \psi_3 \text{ and} \\ \beta_7 &= \psi_5. \end{aligned}$$



Now

$$\begin{aligned}\Delta_1^+ = \{ & \{\psi_1, \psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3, \psi_1 + 2\psi_2 + 2\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}\},\end{aligned}$$

and

$$\begin{aligned}\Delta_2^+ = \{ & \{\psi_6, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \{\psi_5 + \psi_6, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_6 + \psi_7, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}, \{\psi_4 + \psi_5 + \psi_6, \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_5 + \psi_6 + \psi_7, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\}, \{\psi_2 + \psi_4 + \psi_5 + \psi_6, \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_2 + \psi_4 + \psi_5 + \psi_6 + \psi_7\}, \{\psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\}\}\end{aligned}$$

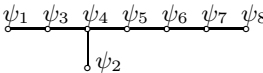
while

$$\begin{aligned}\Delta_4^+ = \{ & \{\psi_4, \psi_2 + \psi_3 + \psi_4 + \psi_5\}, \{\psi_2 + \psi_4, \psi_3 + \psi_4 + \psi_5\}, \\ & \{\psi_3 + \psi_4, \psi_2 + \psi_4 + \psi_5\}, \{\psi_4 + \psi_5, \psi_2 + \psi_3 + \psi_4\}\}\end{aligned}$$

and

$$\Delta_3^+ = \Delta_5^+ = \Delta_6^+ = \Delta_7^+ = \emptyset.$$

In each case if  $\alpha, \alpha' \in \Delta_r^+$  and  $\alpha + \alpha'$  is a root then  $\alpha + \alpha' = \beta_r$ . The assertions of Lemma 6.8 follow by inspection.

Finally suppose that  $\Delta$  is of type  $E_8$  . Then

$$\begin{aligned}\beta_1 &= 2\psi_1 + 3\psi_2 + 4\psi_3 + 6\psi_4 + 5\psi_5 + 4\psi_6 + 3\psi_7 + 2\psi_8, \\ \beta_2 &= 2\psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7, \\ \beta_3 &= \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7, \\ \beta_4 &= \psi_7, \\ \beta_5 &= \psi_2 + \psi_3 + 2\psi_4 + \psi_5, \\ \beta_6 &= \psi_2, \\ \beta_7 &= \psi_3 \text{ and} \\ \beta_8 &= \psi_5.\end{aligned}$$

[illegible]

while

$$\begin{aligned}\Delta_2^+ = \{ & \{\psi_1, \psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3, \psi_1 + 2\psi_2 + 2\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}\},\end{aligned}$$

and

$$\begin{aligned}\Delta_3^+ = \{ & \{\psi_6, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_5 + \psi_6, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_6 + \psi_7, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\}, \\ & \{\psi_4 + \psi_5 + \psi_6, \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_5 + \psi_6 + \psi_7, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\}, \\ & \{\psi_2 + \psi_4 + \psi_5 + \psi_6, \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_2 + \psi_4 + \psi_5 + \psi_6 + \psi_7\}, \\ & \{\psi_4 + \psi_5 + \psi_6 + \psi_7, \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\}\},\end{aligned}$$

and

$$\begin{aligned}\Delta_5^+ = \{ & \{\psi_4, \psi_2 + \psi_3 + \psi_4 + \psi_5\}, \\ & \{\psi_2 + \psi_4, \psi_3 + \psi_4 + \psi_5\}, \\ & \{\psi_3 + \psi_4, \psi_2 + \psi_4 + \psi_5\}, \\ & \{\psi_4 + \psi_5, \psi_2 + \psi_3 + \psi_4\}\},\end{aligned}$$

while

$$\Delta_4^+ = \Delta_6^+ = \Delta_7^+ = \Delta_8^+ = \emptyset.$$

This completes a direct computational verification of the assertions of Lemma 6.8, as opposed to the more direct structural argument in Section 6.

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